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Stability of a pure random delay system with two-time-scale Markovian switching

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ABSTRACT

This work examines almost sure stability of a pure random delay system whose delay time is modeled by a finite state continuous-time Markov chain with two-time scales. The Markov chain contains a fast-varying part and a slowly-changing part. Using the properties of the weighted occupation measure of the Markov chain, it is shown that the overall system's almost-sure-asymptotic stability can be obtained by using the "averaged" delay. This feature implies that even if some longer delay times may destabilize the system individually, the system may still be stable if their impact is balanced. In other words, the Markov chain becomes a stabilizing factor. Numerical results are provided to demonstrate our results.

1. Introduction

Pure time delays occur in a wide range of applications in process control, automotive systems, biomedical sciences, epidemics, transport, communication networks, and population dynamics. Pure delays introduce infinite dimensional systems and affect feedback system stability and performance significantly. Stability analysis of pure delay systems has received considerable and sustained attention

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in the last five decades; see, for example, [2,3,5,12]), among others. For stability of the scalar linear deterministic time-varying pure delay system

$$\dot{x}(t) = -a(t)x\big(t - \gamma(t)\big), \tag{1.1}$$

where $a : [0, \infty) \to (0, \infty)$, $\gamma : [0, \infty) \to [0, q]$ and q is a positive constant, the well-known Yorke's theorem [36, Theorem 1.1] asserts the following result.

Theorem 1.1. *Let $a(t) \leqslant \alpha$ for some $\alpha > 0$.*

(i) *If $\alpha q \leqslant 3/2$, the zero solution of Eq.* (1.1) *is uniformly stable.*
(ii) *If $0 < \alpha q < 3/2$, the zero solution of Eq.* (1.1) *is asymptotically uniformly stable.*

Theorem 1.1 has since been significantly extended. In [32], the authors established the stability theorem under a more general condition. Then in [33], systems with two delays were examined and sufficient conditions on its stability were derived. For more asymptotic properties of these systems, see [8–12,14,16,20,22,23,28,31,34,35]. For the special case of (1.1), the stability of the linear fixed delay system

$$\dot{x}(t) = -ax(t - q), \tag{1.2}$$

is a classical result, in which a and q are positive constants, and the following assertion holds; see also [12].

Theorem 1.2. *The zero solution of Eq.* (1.2) *is asymptotically uniformly stable if and only if $0 < aq < \pi/2$.*

Beyond these results for deterministic systems, random delay systems have recently received increased attention, especially in networked control systems (NCS) that have found broad applications in mobile agents, distributed computing, unmanned aerial and ground vehicles, tele-medicine, smart grids, etc. An NCS is a control system using a dedicated or shared network, often wireless, for communications, control, and coordination among its member of subsystems. While wireless communications provide flexible control structures for system coordination, they also introduce communication latency, data loss, and errors. Due to its mobility, packet routing and signal interferences, communication delays are often random (see [24–26]). Although impact of constant time delays on feedback system stability and performance is well understood [17], characterization of a system's robust stability against random time delays is far more challenging and has drawn significant research effort lately. For example, Huang et al. considered stability of linear uncertain NCSs with random communication time delays which are governed by a continuous-time Markov chain in [13]. By using the generator of a Markov process, [18] and [19] examined mean square stability of discrete-time and continuous-time systems with a Markov-type random delay. By using a Lyapunov–Krasovskii functional, Chen et al. investigated the exponential mean-square stability of an NCS [4]. Although the above papers treated delays in a Markov chain framework, asymptotic properties of the Markov chain are not explicitly explored in the analysis.

This paper studies robust stability of a feedback system against random delays that are modeled by a Markov chain that contains both a fast varying part and a slowly changing part. Consideration of fast varying delays is motivated by control systems with communication channels. Communication latency can occur by many causes, including scheduling delay, multi-hop routing, queueing of packets, signal echoes, among others [21,27]. In particular, packets arrive randomly and their movements through network servers are queued according to their priorities. For example, voice and video signals are often assigned higher priorities than measurement signals and files. If network servers have FIFO (first-in-first-out) queues and a control task is assigned a secondary priority, its packet movement latency depends on higher priority tasks' arrival rates which are random and may be in bursts, leading to random and often fast varying time delays. A salient feature of such delay systems is that the asymptotic properties of the Markov chain, in particular, its stationary distributions become dominant in determining stability of the system. In addition, most existing results include an additional non-delay feedback loop that gives a baseline stability before introducing an additional delay loop. In practical systems, such configurations are often in conflict with physical settings, rendering it necessary to consider pure delay systems, namely all feedback paths are delayed. Mathematically, pure delay systems are much more difficult to treat. To our best knowledge, there is little work in the literature dealing with pure random delay systems.

This paper establishes robust stability conditions for asymptotic stability of the following scalar first-order pure random delay system

$$\dot{x}(t) = f\big(x\big(t - r(t)\big)\big), \tag{1.3}$$

where $r(t) \geqslant 0$ is a continuous-time Markov chain in a finite state space $\mathbb{S} = \{r_1, r_2, \dots, r_m\}$. This system is a random switching system among the following m fixed delay subsystems

$$\dot{x}(t) = f\big(x(t - r_i)\big), \quad i = 1, 2, \dots, m.$$

The random switching is governed by the Markov chain $r(t)$. An interesting question is: If some subsystems are stable and others are unstable, is it possible that the switched system (1.3) is stable? The answers to this question rely on a basic understanding of interaction between the Markov chain and the delay subsystems. To highlight our idea clearly and to show the influence of the continuous-time Markov chain on the asymptotic stability this paper will focus on the system

$$\dot{x}(t) = -ax\big(t - r(t)\big), \tag{1.4}$$

where $a > 0$ is a constant. Thus we revisit the model treated in Yorke's work with a significant extension. That is, the delay now becomes a random process, which leads to a substantial deviation from the classical Yorke's work.

Departing from many of the recent works on random delay systems, we note that the right-hand side of (1.4) does not have a term on the current state $x(t)$, but rather contains only the pure delay term. This creates main difficulties in analysis since there is no reference to the usual exponentially decaying term due to the dependence on $x(t)$. According to Theorem 1.2, for $i = 1, 2, \dots, m$, if $0 < ar_i < \pi/2$, the ith subsystem is stable. This paper aims to find conditions under which the question "If some subsystems are stable and others are not, is it possible that the switched system is still stable?" can be answered affirmatively.

The rest of the paper is arranged as follows. In the next section, we provide motivations of our study, formulate the problem, and define notations. By using convergence properties of weighted occupation measures of a two-time-scale Markov chain, Section 3 establishes the almost sure uniform stability and the almost sure asymptotic uniform stability for the linear, randomly switching pure delay systems. Using the Lipschitz condition and the Yorke condition, Section 4 extends the stability results to the nonlinear random delay systems of functional equations. Section 5 presents numerical experiments to demonstrate our results. Section 6 concludes the paper with further remarks.

2. Preliminary results and motivations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$ satisfying the usual conditions, that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $\tilde{r}(t)$, $t \geqslant 0$, be a continuous-time Markov chain defined on this probability space taking values in a finite state space $\mathbb{S} = \{r_1, r_2, \dots, r_m\}$ with generator $\widetilde{Q} = (\tilde{q}_{ij}) \in \mathbb{R}^{m \times m}$. In this paper, we choose $\mathcal{F}_t = \sigma\{\tilde{r}(s),\ 0 \leqslant s \leqslant t\}$. Without loss of generality, assume that $r_1 < r_2 < \cdots < r_m$. Recall that the generator $\widetilde{Q}$ is weakly irreducible [29], if the system of equations

$$\begin{cases} \tilde{\nu}\widetilde{Q} = 0, \\ \sum_{i=1}^{m} \tilde{\nu}_i = 1 \end{cases} \tag{2.1}$$

has a unique solution $\tilde{\nu} = (\tilde{\nu}_1, \ldots, \tilde{\nu}_m)$ satisfying $\tilde{\nu}_i \geqslant 0$ for each $i = 1, \ldots, m$. The solution $\tilde{\nu} = (\tilde{\nu}_1, \tilde{\nu}_2, \ldots, \tilde{\nu}_m)$ is termed a quasi-stationary distribution. In what follows, we use $O(y)$ to denote the function of y satisfying $\sup_y |O(y)|/|y| < \infty$.

Let the random delay be modeled by a continuous-time Markov chain $\tilde{r}(t)$ with a finite state space $\mathbb{S}$. In our setup, the Markov chain has both fast and slow motions and involves strong and weak interactions. To reflect the fast and slow motions of the Markov chain, we introduce a small parameter $\varepsilon > 0$ and rewrite the Markov chain $\tilde{r}(t)$ as $r^\varepsilon(t)$ and the generator $\widetilde{Q}$ as Q^ε. Then, the Markov chain displays two-time scales, a usual running time t and a stretched (fast) time t/ε. Suppose that the generator of the Markov chain is given by

$$Q^\varepsilon = \frac{Q}{\varepsilon} + Q_0, \tag{2.2}$$

where both Q and Q_0 are generators of suitable continuous-time Markov chains, Q/ε represents the fast-varying part and Q_0 represents the slow-changing part. Throughout the paper, we assume that Q is weakly irreducible in the sense defined in (2.1). Denote the quasi-stationary distribution associated with Q as $\nu = (\nu_1, \ldots, \nu_m)$. As demonstrated in [29, Chapters 4 & 5], using asymptotic expansions, one can show that the asymptotic properties of the probability vector and the transition probability matrix will mainly be determined by Q/ε as $\varepsilon \to 0$. For example, denoting the probability vector by $p^\varepsilon(t) = (\mathbb{P}(r^\varepsilon(t) = r_1), \ldots, \mathbb{P}(r^\varepsilon(t) = r_m))$, then it satisfies the following forward equation $\dot{p}^\varepsilon(t) = p^\varepsilon(t)Q^\varepsilon$ with appropriate initial data $p^\varepsilon(0) = p_0$. One of the results in [29, p. 81] indicates that the solutions of the forward equation can be approximated by means of the asymptotic extensions,

$$\left|p^\varepsilon(t) - \nu\right| \leqslant O\left(\varepsilon + \exp\left(-\frac{\kappa t}{\varepsilon}\right)\right), \tag{2.3}$$

where $\kappa > 0$ is determined by the eigenvalues of Q.

Since the fluctuations of the $r^\varepsilon(t)$ are dominated by the fast-varying part Q/ε, we are interested in the effects of Q/ε on the asymptotic properties of the associated randomly switching systems of differential equations. To highlight the role of Q/ε, let us rewrite system (1.4) as

$$\dot{x}^\varepsilon(t) = -ax^\varepsilon\big(t - r^\varepsilon(t)\big) \tag{2.4}$$

with the initial data $\xi \in C^1([-r_m, 0]; \mathbb{R})$, where $C^1([-r_m, 0]; \mathbb{R})$ denotes the family of continuous and smooth functions from $[-r_m, 0]$ to $\mathbb{R}$ with the norm $\|\varphi\| = \sup_{-r_m \leqslant \theta \leqslant 0} |\varphi(\theta)|$. It is obvious that ξ satisfies the Lipschitz condition on $[-r_m, 0]$, namely, there exists the Lipschitz constant K such that for any $t_1, t_2 \in [-r_m, 0]$, $|\xi(t_1) - \xi(t_2)| \leqslant K|t_1 - t_2|$. It is obvious that 0 is the unique trivial solution (or equilibrium point) of the system (2.4).

Compared to the state x, the Markov chain $r^\varepsilon(\cdot)$ fluctuates much more rapidly and acts essentially as a "noise." We may expect some averaging to take place ("average" effect with respect to its stationary measure). Such average ideas have been explored in [1,6,7,15,30]. In our setup, by "average," we mean the replacement of the random delay $r^\varepsilon(t)$ by its average with respect to the stationary measure $\bar{r} = \sum_{i=1}^m r_i \nu_i$. Motivated by Theorem 1.1 and Theorem 1.2, we ask the question: Under the condition $0 < a\bar{r} < 3/2$ (or $0 < a\bar{r} < \pi/2$), as $\varepsilon \to 0$, can the random delay switching system (2.4) be asymptotic stable? Under some mild conditions, this paper will establish an affirmative answer to this question.

The average condition $0 < a\bar{r} < 3/2$ does not require each individual delay be small. Consequently, it allows some subsystems to be unstable. We shall show that the random delay switching system may be stable even if it switches among stable and unstable subsystems. That is, the continuous-time Markov chain may be a stabilizing factor. We now define the notion of stability used in this paper.

Definition 2.1. The trivial solution of the random delay system (2.4) is said to be

(i) *almost surely uniformly stable as* $\varepsilon \to 0$ if for any $\eta > 0$, there exists a $\delta = \delta(\eta) \in (0, \eta]$ such that for any initial data $\xi \in C^1([-r_m, 0]; \mathbb{R})$ with $\|\xi\| \leqslant \delta$ and all $t \geqslant 0$,

$$\lim_{\varepsilon \to 0} \left| x^\varepsilon(t) \right| \leqslant \eta, \quad \text{a.s.};$$

(ii) *almost surely asymptotically uniformly stable as* $\varepsilon \to 0$ if it is almost surely uniformly stable as $\varepsilon \to 0$ and for η defined by (i), there exists $T_0 = T_0(\eta) > 0$ such that for all $t \geqslant T_0$, $\|\xi\| < \delta$,

$$\lim_{\varepsilon \to 0} \left| x^\varepsilon(t) \right| \leqslant \eta, \quad \text{a.s.}$$

3. Almost surely uniform stability and asymptotic uniform stability

In this section, using the convergence of the weighted occupation measure of $r^\varepsilon(t)$ as $\varepsilon \to 0$ as the bridge, we establish the almost sure uniform stability and the almost sure asymptotic stability theorems for the random delay system (2.4). The main motivating force is the work [36], which is one of the most important stability results for deterministic delay systems, especially for pure delay systems (see also [10,12,14,31–35]).

Theorem 3.1. *Suppose that one of the following conditions is satisfied*:

(i) $a\bar{r} \leqslant 1$;
(ii) $r_1, \ldots, r_j < \frac{1}{a}$, $r_{j+1}, \ldots, r_m \geqslant \frac{1}{a}$, $\frac{a^2}{2} \sum_{i=1}^{j} \nu_i r_i^2 + a \sum_{i=j+1}^{m} \nu_i (r_i - \frac{1}{2a}) \leqslant 1$;
(iii) *all* $r_1, \ldots, r_m \geqslant \frac{1}{a}$, $a\bar{r} \leqslant \frac{3}{2}$.

Then the trivial solution of (2.4) *is almost surely uniformly stable as* $\varepsilon \to 0$.

Theorem 3.2. *Suppose that one of the following conditions is satisfied*:

(i) $0 < a\bar{r} < 1$;
(ii) $r_1, \ldots, r_j < \frac{1}{a}$, $r_{j+1}, \ldots, r_m \geqslant \frac{1}{a}$, $0 < \frac{a^2}{2} \sum_{i=1}^{j} \nu_i r_i^2 + a \sum_{i=j+1}^{m} \nu_i (r_i - \frac{1}{2a}) < 1$;
(iii) *all* $r_1, \ldots, r_m \geqslant \frac{1}{a}$, $0 < a\bar{r} < \frac{3}{2}$.

Then the trivial solution of (2.4) *is almost surely asymptotically uniformly stable as* $\varepsilon \to 0$.

Remark 3.3. Although each of the conditions in (i), (ii), and (iii) is a little stronger than the expected condition “$0 < a\bar{r} < 3/2$,” they show that the Markov chain may be a stabilizing factor. It need not require for every $r_i \in \mathbb{S}$, $0 < ar_i < 3/2$. That is, it need not be assumed that all subsystems are stable.

We also note that in the above, (ii) implies (iii) in the following sense. If all $r_j \geqslant (1/a)$, then the fist summation part in (ii) becomes 0, and the last summation leads to exactly (iii).

The proofs of the two theorems are technical, and hence are divided into a sequence of lemmas. We proceed with presenting these lemmas.

Lemma 3.4. *For any initial data* $\xi \in C^1([-r_m, 0]; \mathbb{R})$ *and* $T > 0$, *the solution* $x(t)$ *of the system* (2.4) *holds the property*

$$\sup_{-r_m \leqslant t \leqslant T} \left| x^\varepsilon(t) \right| \leqslant 2\|\xi\| e^{aT}.$$

Proof. We rewrite (2.4) as

$$x^{\varepsilon}(t) = x^{\varepsilon}(0) - a\int_0^t x^{\varepsilon}\big(s - r^{\varepsilon}(s)\big)\,ds.$$

Note that

$$\big|x^{\varepsilon}(t)\big| \leqslant \big|\xi(0)\big| + a\int_0^t \big|x^{\varepsilon}\big(s - r^{\varepsilon}(s)\big)\big|\,ds.$$

For any $t_1 \in (0, T]$, noting that $0 \leqslant r^{\varepsilon}(t) \leqslant r_m$, we have

$$\sup_{0\leqslant t\leqslant t_1}\big|x^{\varepsilon}(t)\big| \leqslant \|\xi\| + a\int_0^{t_1}\Big(\sup_{-r_m\leqslant v\leqslant s}\big|x^{\varepsilon}(v)\big|\Big)\,ds.$$

It follows that

$$\begin{aligned}\sup_{-r_m\leqslant t\leqslant t_1}\big|x^{\varepsilon}(t)\big| &\leqslant \sup_{-r_m\leqslant t\leqslant 0}\big|x^{\varepsilon}(t)\big| + \sup_{0\leqslant t\leqslant t_1}\big|x^{\varepsilon}(t)\big| \\ &\leqslant 2\|\xi\| + a\int_0^{t_1}\Big(\sup_{-r_m\leqslant v\leqslant s}\big|x^{\varepsilon}(v)\big|\Big)\,ds.\end{aligned}$$

Applying the Gronwall inequality yields

$$\sup_{-r_m\leqslant t\leqslant t_1}\big|x^{\varepsilon}(t)\big| \leqslant 2\|\xi\|e^{at_1} \leqslant 2\|\xi\|e^{aT}.$$

Choosing $t_1 = T$ gives the desired assertion. □

Let $x^{\varepsilon}(t)$ be the solution of the system (2.4). For $r_i \in \mathbb{S}$ and any $t \in [0, T]$, define a sequence of weighted occupation measures $Z_i^{\varepsilon}(t)$ as

$$Z_i^{\varepsilon}(t) = \int_0^t x^{\varepsilon}(s - r_i)[I_{\{r^{\varepsilon}(s)=r_i\}} - \nu_i]\,ds, \tag{3.1}$$

where I_A is the indicator function of the set A. Denote $Z^{\varepsilon}(t) = (Z_1^{\varepsilon}(t), \ldots, Z_m^{\varepsilon}(t))'$. It is a measure of the functional occupancy for the process $r^{\varepsilon}(\cdot)$. The following lemma illustrates the convergence of the weighted occupation measure of $r^{\varepsilon}(t)$ as $\varepsilon \to 0$.

Lemma 3.5. *Let $Z_i^{\varepsilon}(t)$ be defined by* (3.1)*. Then*

$$\lim_{\varepsilon\to 0}\Big(\sup_{0\leqslant t\leqslant T}\mathbb{E}\big|Z_i^{\varepsilon}(t)\big|^2\Big) = 0. \tag{3.2}$$

Proof. For any $\zeta \in (0,1)$ and $t \in (0,T]$, let $N = \lfloor t/\varepsilon^{1-\zeta} \rfloor$, where $\lfloor a \rfloor$ denotes the integer part of a. Use a partition of $[0,t]$ given by

$$[0,t] = [t_0, t_1] \cup [t_1, t_2) \cup [t_2, t_3) \cup \cdots \cup [t_N, t_{N+1}],$$

where $t_k = \varepsilon^{1-\zeta} k$ for $k = 0, 1, \ldots, N$ and $t_{N+1} = t$. Define a piecewise-constant function

$$\tilde{x}^\varepsilon(t - r_i) = \begin{cases} x^\varepsilon(-r_i), & \text{if } 0 \leqslant t < t_2, \\ x^\varepsilon(t_{k-1} - r_i), & \text{if } t_k \leqslant t < t_{k+1}, \\ x^\varepsilon(t_{N-1} - r_i), & \text{if } t = t_{N+1}. \end{cases}$$

In view of the elementary inequality $(a+b)^2 \leqslant 2a^2 + 2b^2$,

$$\begin{aligned} \mathbb{E}\big|Z_i^\varepsilon(t)\big|^2 \leqslant{}& 2\mathbb{E}\left| \int_0^t \big[x^\varepsilon(s - r_i) - \tilde{x}^\varepsilon(s - r_i)\big][I_{\{r^\varepsilon(s) = r_i\}} - \nu_i]\,ds \right|^2 \\ &+ 2\mathbb{E}\left| \int_0^t \tilde{x}^\varepsilon(s - r_i)[I_{\{r^\varepsilon(s) = r_i\}} - \nu_i]\,ds \right|^2. \end{aligned} \tag{3.3}$$

Noting $|I_{\{r^\varepsilon(s)=r_i\}} - \nu_i| \leqslant 1$, by virtue of the Cauchy–Schwarz inequality, for any $t \in [0,T]$,

$$\begin{aligned} &\mathbb{E}\left| \int_0^t \big[x^\varepsilon(s - r_i) - \tilde{x}^\varepsilon(s - r_i)\big][I_{\{r^\varepsilon(s) = r_i\}} - \nu_i]\,ds \right|^2 \\ &\quad\leqslant T\mathbb{E}\int_0^t \big[x^\varepsilon(s - r_i) - \tilde{x}^\varepsilon(s - r_i)\big]^2\,ds \\ &\quad= T\int_0^t \mathbb{E}\big[x^\varepsilon(s - r_i) - \tilde{x}^\varepsilon(s - r_i)\big]^2\,ds. \end{aligned}$$

For any $\tau \in [0, t_2]$,

$$\begin{aligned} \int_0^\tau \mathbb{E}\big[x^\varepsilon(s - r_i) - \tilde{x}^\varepsilon(s - r_i)\big]^2\,ds &= \int_0^\tau \mathbb{E}\big[x^\varepsilon(s - r_i) - \tilde{x}^\varepsilon(-r_i)\big]^2\,ds \\ &\leqslant \int_0^{t_2} \mathbb{E}\big[x^\varepsilon(s - r_i) - \xi(-r_i)\big]^2\,ds. \end{aligned}$$

If $t_2 - r_i \leqslant 0$, then for any $s \in [0, t_2]$, $x^\varepsilon(s - r_i) = \xi(s - r_i)$. Since $\xi \in C^1([-r_m, 0]; \mathbb{R})$ satisfies the Lipschitz condition, we have

$$\int_0^{t_2} \mathbb{E}\big[x^\varepsilon(s - r_i) - \xi(-r_i)\big]^2\,ds \leqslant K^2 \int_0^{t_2} s^2\,ds = \frac{K^2}{3} t_2^3 = O\big(\varepsilon^{3-3\zeta}\big), \tag{3.4}$$

where K is the Lipschitz constant. When $t_2 - r_i > 0$, by the computation of (3.4),

$$\begin{aligned}
&\int_0^{t_2} \mathbb{E}\big[x^\varepsilon(s-r_i) - \xi(-r_i)\big]^2\,ds \\
&= \int_0^{r_i} \mathbb{E}\big[\xi(s-r_i) - \xi(-r_i)\big]^2\,ds + \int_{r_i}^{t_2} \mathbb{E}\big[x^\varepsilon(s-r_i) - \xi(-r_i)\big]^2\,ds \\
&\leqslant K^2 \int_0^{r_i} s^2\,ds + 2\int_{r_i}^{t_2} \mathbb{E}\big|x^\varepsilon(s-r_i) - x^\varepsilon(0)\big|^2\,ds + 2\int_{r_i}^{t_2} \mathbb{E}\big[x^\varepsilon(0) - \xi(-r_i)\big]^2\,ds \\
&= \frac{K^2}{3} r_i^3 + 2\int_{r_i}^{t_2} \mathbb{E}\big|x^\varepsilon(s-r_i) - x^\varepsilon(0)\big|^2\,ds + 2K^2 r_i^2 (t_2 - r_i) \\
&= O\big(\varepsilon^{3-3\zeta}\big) + 2\int_{r_i}^{t_2} \mathbb{E}\big|x^\varepsilon(s-r_i) - x^\varepsilon(0)\big|^2\,ds.
\end{aligned}$$

Noting that $s - r_i \geqslant 0$ for $s \in [r_i, t_2]$, by Lemma 3.4,

$$\begin{aligned}
\mathbb{E}\big|x^\varepsilon(s-r_i) - x^\varepsilon(0)\big|^2 &= \mathbb{E}\left|\int_0^{s-r_i} \dot{x}^\varepsilon(\varsigma)\,d\varsigma\right|^2 \\
&= \mathbb{E}\left|-a\int_0^{s-r_i} x^\varepsilon\big(\varsigma - r^\varepsilon(\varsigma)\big)\,d\varsigma\right|^2 \\
&\leqslant 4a^2\|\xi\|^2 e^{2aT}(s-r_i)^2.
\end{aligned}$$

It therefore follows that

$$\begin{aligned}
\int_{r_i}^{t_2} \mathbb{E}\big|x^\varepsilon(s-r_i) - x^\varepsilon(0)\big|^2\,ds &\leqslant 4a^2\|\xi\|^2 e^{2aT} \int_{r_i}^{t_2} (s-r_i)^2\,ds \\
&\leqslant \frac{4}{3}a^2\|\xi\|^2 e^{2aT}(t_2 - r_i)^3 \\
&= O\big(\varepsilon^{3-3\zeta}\big),
\end{aligned}$$

which shows that when $t_2 - r_i > 0$,

$$\int_0^{t_2} \mathbb{E}\big[x^\varepsilon(s-r_i) - \xi(-r_i)\big]^2\,ds = O\big(\varepsilon^{3-3\zeta}\big). \tag{3.5}$$

This together with (3.4) yields

$$\int_0^{t_2} \mathbb{E}\big[x^\varepsilon(s-r_i)-\tilde{x}^\varepsilon(s-r_i)\big]^2\,ds = O\big(\varepsilon^{3-3\zeta}\big).$$

It follows that

$$\int_0^t \mathbb{E}\big|x^\varepsilon(s-r_i)-\tilde{x}^\varepsilon(s-r_i)\big|^2\,ds = \sum_{k=2}^N \int_{t_k}^{t_{k+1}} \mathbb{E}\big|x^\varepsilon(s-r_i)-x^\varepsilon(t_{k-1}-r_i)\big|^2\,ds + O\big(\varepsilon^{3-3\zeta}\big).$$

Now let us consider

$$\int_{t_k}^{t_{k+1}} \mathbb{E}\big|x^\varepsilon(s-r_i)-x^\varepsilon(t_{k-1}-r_i)\big|^2\,ds.$$

If $t_{k-1}-r_i>0$, by Lemma 3.4,

$$\begin{aligned}
&\int_{t_k}^{t_{k+1}} \mathbb{E}\big|x^\varepsilon(s-r_i)-x^\varepsilon(t_{k-1}-r_i)\big|^2\,ds \\
&\quad= \int_{t_k}^{t_{k+1}} \mathbb{E}\left|\int_{t_{k-1}-r_i}^{s-r_i} \dot{x}^\varepsilon(\varsigma)\,d\varsigma\right|^2 ds = a^2\int_{t_k}^{t_{k+1}} \mathbb{E}\left|\int_{t_{k-1}-r_i}^{s-r_i} x^\varepsilon(\varsigma)\,d\varsigma\right|^2 ds \\
&\quad\leqslant 4a^2\|\xi\|^2 e^{2aT}\int_{t_k}^{t_{k+1}} (s-t_{k-1})^2\,ds \\
&\quad\leqslant \frac{4}{3}a^2\|\xi\|^2 e^{2aT}(t_{k+1}-t_{k-1})^3 = O\big(\varepsilon^{3-3\zeta}\big).
\end{aligned}$$

If $t_{k+1}-r_i\leqslant 0$, similar to (3.4),

$$\int_{t_k}^{t_{k+1}} \mathbb{E}\big|x^\varepsilon(s-r_i)-x^\varepsilon(t_{k-1}-r_i)\big|^2\,ds = O\big(\varepsilon^{3-3\zeta}\big).$$

If $t_{k-1}-r_i\leqslant 0\leqslant t_{k+1}-r_i$, similar to (3.5),

$$\int_{t_k}^{t_{k+1}} \mathbb{E}\big|x^\varepsilon(s-r_i)-x^\varepsilon(t_{k-1}-r_i)\big|^2\,ds = O\big(\varepsilon^{3-3\zeta}\big).$$

Noting that $t=N\varepsilon^{1-\zeta}+(t_{N+1}-t_N)\leqslant T$, we have

$$\int_0^t \mathbb{E}\big[x^\varepsilon(s-r_i)-\tilde{x}^\varepsilon(s-r_i)\big]^2\,ds = \sum_{k=0}^N O\big(\varepsilon^{3-3\zeta}\big) = O\big(\varepsilon^{2-2\zeta}\big). \tag{3.6}$$

Let us now estimate the second term of (3.3). Denote

$$\tilde{\eta}^{\varepsilon}(t)=\mathbb{E}\left[\int_0^t \tilde{x}^{\varepsilon}(s-r_i)(I_{\{r^{\varepsilon}(s)=r_i\}}-\nu_i)\,ds\right]^2.$$

Note that for any $t\in[0,T]$, $|\tilde{x}^{\varepsilon}(t-r_i)(I_{\{r^{\varepsilon}(t)=r_i\}}-\nu_i)|$ is bounded. Then the derivative of $\tilde{\eta}^{\varepsilon}(t)$ is given by

$$\frac{d\tilde{\eta}^{\varepsilon}(t)}{dt}=2\mathbb{E}\int_0^t \tilde{x}^{\varepsilon}(t-r_i)\tilde{x}^{\varepsilon}(s-r_i)(I_{\{r^{\varepsilon}(t)=r_i\}}-\nu_i)(I_{\{r^{\varepsilon}(s)=r_i\}}-\nu_i)\,ds.$$

Since $\tilde{x}^{\varepsilon}(t-r_i)\tilde{x}^{\varepsilon}(s-r_i)(I_{\{r^{\varepsilon}(t)=r_i\}}-\nu_i)(I_{\{r^{\varepsilon}(s)=r_i\}}-\nu_i)$ is bounded for any $s,t\in[0,T]$, for any $\tau\in[0,t_2]$, there must exist a constant K_1 such that

$$\mathbb{E}\int_0^{\tau} \tilde{x}^{\varepsilon}(t-r_i)\tilde{x}^{\varepsilon}(s-r_i)(I_{\{r^{\varepsilon}(t)=r_i\}}-\nu_i)(I_{\{r^{\varepsilon}(s)=r_i\}}-\nu_i)\,ds=K_1\tau\leqslant K_1t_2=O\left(\varepsilon^{1-\zeta}\right).$$

If $\tau\in[t_k,t_{k+1})$ for $k=2,3,\dots,N$, then using the same argument,

$$\mathbb{E}\int_{t_{k-1}}^{\tau} \tilde{x}^{\varepsilon}(t-r_i)\tilde{x}^{\varepsilon}(s-r_i)(I_{\{r^{\varepsilon}(t)=r_i\}}-\nu_i)(I_{\{r^{\varepsilon}(s)=r_i\}}-\nu_i)\,ds=O\left(\varepsilon^{1-\zeta}\right).$$

Hence we have

$$\frac{d\tilde{\eta}^{\varepsilon}(t)}{dt}=2\int_0^{t_{k-1}} \mathbb{E}\big(\tilde{x}^{\varepsilon}(t-r_i)\tilde{x}^{\varepsilon}(s-r_i)[I_{\{r^{\varepsilon}(t)=r_i\}}-\nu_i][I_{\{r^{\varepsilon}(s)=r_i\}}-\nu_i]\big)\,ds+O\left(\varepsilon^{1-\zeta}\right). \tag{3.7}$$

Recall that $\mathcal{F}_t^{\varepsilon}=\sigma\{r^{\varepsilon}(s),\ 0\leqslant s\leqslant t\}$. For any $s_{k-1}<s\leqslant t_{k-1}<t_k\leqslant t<t_{k+1}$, using the asymptotic expansion of the probability vector of $r^{\varepsilon}(t)$ (see [29, Lemma 5.1, p. 81]), there exists a constant κ such that

$$\begin{aligned}
&\mathbb{E}\big(\tilde{x}^{\varepsilon}(t-r_i)\tilde{x}^{\varepsilon}(s-r_i)[I_{\{r^{\varepsilon}(t)=r_i\}}-\nu_i][I_{\{r^{\varepsilon}(s)=r_i\}}-\nu_i]\big)\\
&\quad=\mathbb{E}\big(\tilde{x}^{\varepsilon}(t-r_i)\tilde{x}^{\varepsilon}(s-r_i)[I_{\{r^{\varepsilon}(s)=r_i\}}-\nu_i]\mathbb{E}\big[(I_{\{r^{\varepsilon}(t)=r_i\}}-\nu_i)|\mathcal{F}_{t_{k-1}}\big]\big)\\
&\quad=\mathbb{E}\big[\tilde{x}^{\varepsilon}(t-r_i)\tilde{x}^{\varepsilon}(s-r_i)[I_{\{r^{\varepsilon}(s)=r_i\}}-\nu_i]O\left(\varepsilon+e^{-\frac{\kappa(t-t_{k-1})}{\varepsilon}}\right)\big]\\
&\quad=\mathbb{E}\big[\tilde{x}^{\varepsilon}(t-r_i)\tilde{x}^{\varepsilon}(s-r_i)[I_{\{r^{\varepsilon}(s)=r_i\}}-\nu_i]O\left(\varepsilon+e^{-\frac{\kappa(t_k-t_{k-1})}{\varepsilon}}\right)\big]\\
&\quad=\mathbb{E}\big[\tilde{x}^{\varepsilon}(t-r_i)\tilde{x}^{\varepsilon}(s-r_i)[I_{\{r^{\varepsilon}(s)=r_i\}}-\nu_i]\big]O\left(\varepsilon+e^{-\frac{\kappa}{\varepsilon^{\zeta}}}\right)\\
&\quad=\mathbb{E}\big[\tilde{x}^{\varepsilon}(t-r_i)\tilde{x}^{\varepsilon}(s-r_i)[I_{\{r^{\varepsilon}(s)=r_i\}}-\nu_i]\big]O(\varepsilon)\\
&\quad=O(\varepsilon).
\end{aligned}$$

This together with (3.7) gives

$$\frac{d\tilde{\eta}^{\varepsilon}(t)}{dt} = O\left(\varepsilon^{1-\zeta}\right), \tag{3.8}$$

which holds uniformly on $[0, T]$. Eq. (3.8) together with $\tilde{\eta}^{\varepsilon}(0) = 0$ yields

$$\sup_{0\leqslant t\leqslant T} \tilde{\eta}^{\varepsilon}(t) = \sup_{0\leqslant t\leqslant T} \int_0^t \left(\frac{d\tilde{\eta}(\varsigma)}{d\varsigma}\right) d\varsigma = O\left(\varepsilon^{1-\zeta}\right). \tag{3.9}$$

Combining (3.6) and (3.9) gives $\mathbb{E}|Z_i^{\varepsilon}(t)|^2 = O(\varepsilon^{1-\zeta})$, which implies (3.2), as required. □

Applying Lemmas 3.4 and 3.5 yields the following convergence of $Z_i(t)$.

Lemma 3.6. *Let Z_i^{ε} be defined by* (3.1)*. For any $t \in [0, T]$,*

$$\mathbb{P}\left(\lim_{\varepsilon\to 0}\left|Z_i^{\varepsilon}(t)\right| = 0\right) = 1.$$

Proof. According to Lemma 3.5, for any $t \in [0, T]$,

$$\lim_{\varepsilon\to 0} \mathbb{E}\left|Z_i^{\varepsilon}(t)\right|^2 = 0. \tag{3.10}$$

By Lemma 3.4, $\sup_{-r_m\leqslant t\leqslant T} |x^{\varepsilon}(t)| \leqslant 2\|\xi\|e^{aT}$. By the definition of $Z_i^{\varepsilon}(t)$, for any $t \in [0, T]$ and $\varepsilon > 0$, $|Z_i^{\varepsilon}(t)|$ is uniformly bounded, where the bounding constant may depend on T and $\|\xi\|$. We claim that for any $t \in [0, T]$, $\lim_{\varepsilon\to 0} |Z_i^{\varepsilon}(t)|$ exists almost surely. For suppose not, there would exist two convergent subsequences with distinct limits. That is, there exist ε_1 and $\varepsilon_2 > 0$ such that $\lim_{\varepsilon_1\to 0} |Z_i^{\varepsilon_1}(t)| \neq \lim_{\varepsilon_2\to 0} |Z_i^{\varepsilon_2}(t)|$ a.s., which implies that

$$\eta = \left|\lim_{\varepsilon_1\to 0}\left|Z_i^{\varepsilon_1}(t)\right|^2 - \lim_{\varepsilon_2\to 0}\left|Z_i^{\varepsilon_2}(t)\right|^2\right| > 0 \quad \text{a.s.}$$

since for $x \geqslant 0$, $f(x) = x^2$ is a monotonic function. Note that

$$\mathbb{E}\eta \leqslant \mathbb{E}\left(\lim_{\varepsilon_1\to 0}\left|Z_i^{\varepsilon_1}(t)\right|^2\right) + \mathbb{E}\left(\lim_{\varepsilon_2\to 0}\left|Z_i^{\varepsilon_2}(t)\right|^2\right).$$

By (3.10), for any $t \in [0, T]$ and each ε_j $(j = 1, 2)$,

$$\lim_{\varepsilon_j\to 0} \mathbb{E}\left|Z_i^{\varepsilon_j}(t)\right|^2 = 0. \tag{3.11}$$

Noting that $\lim_{\varepsilon_j\to 0} |Z_i^{\varepsilon_j}(t)|$ exists (so $\liminf = \lim$), by Fatou's Lemma, for any $t \in [0, T]$,

$$\mathbb{E}\left(\lim_{\varepsilon_j\to 0}\left|Z_i^{\varepsilon_j}(t)\right|^2\right) = \mathbb{E}\left(\liminf_{\varepsilon_j\to 0}\left|Z_i^{\varepsilon_j}(t)\right|^2\right) \leqslant \liminf_{\varepsilon_j\to 0}\mathbb{E}\left|Z_i^{\varepsilon_j}(t)\right|^2 = \lim_{\varepsilon_j\to 0}\mathbb{E}\left|Z_i^{\varepsilon_j}(t)\right|^2 = 0. \tag{3.12}$$

This implies that $\mathbb{E}\eta \leqslant 0$, which is a contradiction to $\eta > 0$ a.s. This shows that $\lim_{\varepsilon\to 0} |Z_i^{\varepsilon}(t)|$ exists almost surely.

By (3.10), for any $t \in [0, T]$, applying Fatou's Lemma again gives

$$\mathbb{E}\Big(\lim_{\varepsilon\to 0}\big|Z^{\varepsilon}(t)\big|^2\Big) \leqslant \lim_{\varepsilon\to 0}\mathbb{E}\big|Z^{\varepsilon}(t)\big|^2 = 0. \tag{3.13}$$

For any $t \in [0, T]$, we have

$$\lim_{\varepsilon\to 0}\big|Z^{\varepsilon}(t)\big|^2 = \Big(\lim_{\varepsilon\to 0}\big|Z^{\varepsilon}(t)\big|\Big)^2.$$

This, together with (3.13) gives

$$\mathbb{E}\Big(\lim_{\varepsilon\to 0}\big|Z^{\varepsilon}(t)\big|\Big)^2 = 0. \tag{3.14}$$

Let $X(t) = \lim_{\varepsilon\to 0}|Z^{\varepsilon}(t)|$. Note that for any $t \in [0, T]$, $X(t) \geqslant 0$, so (3.14) yields $\mathbb{E}X^2(t) = 0$. This implies that $X(t) = 0$ a.s. That is, for any $t \in [0, T]$,

$$\mathbb{P}\Big(\lim_{\varepsilon\to 0}\big|Z^{\varepsilon}(t)\big| = 0\Big) = 1,$$

which completes the proof of the lemma. □

Lemma 3.7. *For some $t_1 \geqslant 0$, let $x^{\varepsilon}(t)$ be a solution of (2.4) on $[t_1 - r_m, t_1]$. If $x^{\varepsilon}(t) \neq 0$ for all $t \in [t_1 - r_m, t_1]$, then*

$$x^{\varepsilon}(t_1)\dot{x}^{\varepsilon}(t)|_{t=t_1} < 0.$$

Proof. Note that all $r_i \leqslant r_m$, $i = 1, \ldots, m$, and that $x^{\varepsilon}(t) \neq 0$ for all $t \in [t_1 - r_m, t_1]$, which imply that $x^{\varepsilon}(t)$ does not change sign in $[t_1 - r_m, t_1]$. We therefore have

$$x^{\varepsilon}(t_1)x^{\varepsilon}\big(t_1 - r^{\varepsilon}(t_1)\big) > 0.$$

This leads to

$$x^{\varepsilon}(t_1)\dot{x}^{\varepsilon}(t)|_{t=t_1} = -ax^{\varepsilon}(t_1)x^{\varepsilon}\big(t_1 - r^{\varepsilon}(t_1)\big) < 0,$$

as required. □

Lemma 3.8. *Assume that the conditions in Theorem 3.1 are satisfied and let $t_1 \geqslant r_m$. Let $x^{\varepsilon}(t)$ be a solution of (2.4) on $[t_1 - 2r_m, T]$ such that $T > t_1 + r_m$ and $x^{\varepsilon}(t_1) = 0$. Then*

$$\mathbb{P}\Big[\sup_{t\in[t_1,T]}\Big(\lim_{\varepsilon\to 0}\big|x^{\varepsilon}(t)\big|\Big) \leqslant \sup_{s\in[t_1-2r_m,t_1]}\Big(\lim_{\varepsilon\to 0}\big|x^{\varepsilon}(s)\big|\Big)\Big] = 1. \tag{3.15}$$

Proof. The proof is divided into two steps.

Step 1: Preliminary estimates. Suppose that this assertion were not true. Then there would exist $A \subset \Omega$ with $\mathbb{P}(A) > 0$ such that for each $\omega \in A$,

$$\sup_{t\in[t_1,T]}\Big(\lim_{\varepsilon\to 0}\big|x^{\varepsilon}(\omega, t)\big|\Big) > \sup_{t\in[t_1-2r_m,t_1]}\Big(\lim_{\varepsilon\to 0}\big|x^{\varepsilon}(\omega, t)\big|\Big). \tag{3.16}$$

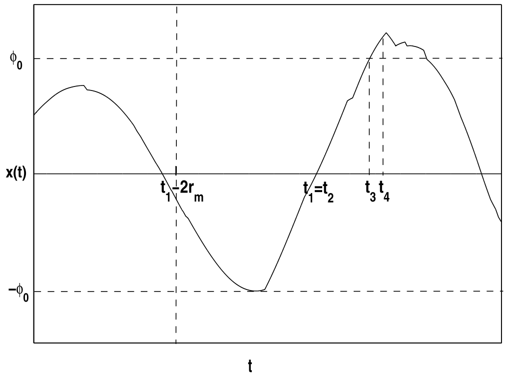

Fig. 1. Choice of t_1, t_2, t_3 and t_4.

Lemma 3.4 yields that $x^\varepsilon(t)$ is uniformly bounded on $[-r_m, T]$, which implies that (3.16) is equivalent to

$$\lim_{\varepsilon\to 0}\Bigl(\sup_{t\in[t_1,T]}\bigl|x^\varepsilon(\omega,t)\bigr|\Bigr) > \lim_{\varepsilon\to 0}\Bigl(\sup_{t\in[t_1-2r_m,t_1]}\bigl|x^\varepsilon(\omega,t)\bigr|\Bigr), \tag{3.17}$$

which implies that there exists a sufficiently small ε_0 such that for any $\varepsilon \in (0, \varepsilon_0]$,

$$\sup_{t\in[t_1,T]}\bigl|x^\varepsilon(\omega,t)\bigr| > \sup_{t\in[t_1-2r_m,t_1]}\bigl|x^\varepsilon(\omega,t)\bigr|. \tag{3.18}$$

Let us fix $\omega_0 \in A$ and let $\phi_0 = \sup_{s\in[t_1-2r_m,t_1]}|x^\varepsilon(\omega_0,t)|$. Noting that $x^\varepsilon(t_1) = 0$, (3.18) implies that there exist

$$t_3 = \inf\bigl\{t > t_1;\ \bigl|x^\varepsilon(\omega_0,t)\bigr| > \phi_0\bigr\} \quad \text{and} \quad t_2 = \sup\bigl\{t < t_3;\ \bigl|x^\varepsilon(\omega_0,t)\bigr| = 0\bigr\}.$$

Then by continuity of the solution, $|x^\varepsilon(\omega_0,t_3)| = \phi_0$ and

$$t_1 \leqslant t_2 < t_3 < T.$$

Note that for all $t \in (t_2,t_3]$, $x^\varepsilon(\omega_0,t) \neq 0$. We suppose that $x^\varepsilon(\omega_0,t) > 0$ for all $t \in (t_2,t_3]$ (the proof for the case $x^\varepsilon(\omega_0,t) < 0$ is similar). Then from the definition of t_3, there exists $t_4 > t_3$ such that $\dot{x}^\varepsilon(\omega_0,t)|_{t=t_4} > 0$ and $x^\varepsilon(\omega_0,t) > \phi_0$ for all $t \in (t_3,t_4]$ and $x^\varepsilon(\omega_0,t_4) = \sup_{s\in[t_3,t_4]} x^\varepsilon(\omega_0,s)$. Fig. 1 illustrates the choice of t_1, t_2, t_3, and t_4. It follows from Lemma 3.7 that $t_4 < t_2 + r_m$, where we assumed that $t_2 + r_m < T$ since T is an arbitrary constant. Let

$$\phi = \sup_{s\in[t_3,t_4]} x^\varepsilon(\omega_0,s) = x^\varepsilon(\omega_0,t_4). \tag{3.19}$$

It follows that for all $t \in [t_1 - r_m, t_4]$,

$$\bigl|\dot{x}^\varepsilon(\omega_0,t)\bigr| \leqslant a\bigl|x^\varepsilon\bigl(\omega_0,t-r^\varepsilon(t)\bigr)\bigr| \leqslant a\phi \tag{3.20}$$

and

$$\begin{aligned}\left|x^{\varepsilon}(\omega_0,t)\right| &= \left|x^{\varepsilon}(\omega_0,t_2)-x^{\varepsilon}(\omega_0,t)\right| = \left|\int_t^{t_2}\dot{x}^{\varepsilon}(\omega_0,s)\,ds\right| \\ &= a\left|\int_t^{t_2}x^{\varepsilon}\big(\omega_0,s-r^{\varepsilon}(s)\big)\,ds\right| \leqslant a\phi|t_2-t|, \qquad (3.21)\end{aligned}$$

where we used the condition $x^{\varepsilon}(\omega_0,t_2)=0$. It follows from (3.20) and (3.21) that for $t_2\leqslant t+t_2\leqslant \min\{t_2+r_1,t_4\}$,

$$\begin{aligned}&\dot{x}^{\varepsilon}(\omega_0,t+t_2)\\ &= -a\sum_{i=1}^m x^{\varepsilon}(\omega_0,t+t_2-r_i)I_{\{r^{\varepsilon}(t+t_2)=r_i\}}\\ &= -a\sum_{i=1}^m x^{\varepsilon}(\omega_0,t+t_2-r_i)[I_{\{r^{\varepsilon}(t+t_2)=r_i\}}-\nu_i]-a\sum_{i=1}^m\nu_i x^{\varepsilon}(\omega_0,t+t_2-r_i)\\ &\leqslant -a\sum_{i=1}^m x^{\varepsilon}(\omega_0,t+t_2-r_i)[I_{\{r^{\varepsilon}(t+t_2)=r_i\}}-\nu_i]+a\sum_{i=1}^m\nu_i\left|x^{\varepsilon}(\omega_0,t+t_2-r_i)\right|\\ &\leqslant -a\sum_{i=1}^m x^{\varepsilon}(\omega_0,t+t_2-r_i)[I_{\{r^{\varepsilon}(t+t_2)=r_i\}}-\nu_i]+a\sum_{i=1}^m\nu_i \min\big\{\phi,a\phi(r_i-t)\big\}\\ &= -a\sum_{i=1}^m x^{\varepsilon}(\omega_0,t+t_2-r_i)[I_{\{r^{\varepsilon}(t+t_2)=r_i\}}-\nu_i]+a\phi\sum_{i=1}^m\nu_i \min\big\{1,a(r_i-t)\big\}. \qquad (3.22)\end{aligned}$$

Note that $t\in[r_1,t_4-t_2]$ is equivalent to $t+t_2-r_1\in[t_2,t_4-r_1]$, so $x^{\varepsilon}(\omega_0,t+t_2-r_1)\geqslant 0$. We therefore have that for all $t\in[r_1,r_2)$,

$$\begin{aligned}&\dot{x}^{\varepsilon}(\omega_0,t+t_2)\\ &= -a\sum_{i=1}^m x^{\varepsilon}(\omega_0,t+t_2-r_i)[I_{\{r^{\varepsilon}(t+t_2)=r_i\}}-\nu_i]-a\sum_{i=1}^m\nu_i x^{\varepsilon}(\omega_0,t+t_2-r_i)\\ &\leqslant -a\sum_{i=1}^m x^{\varepsilon}(\omega_0,t+t_2-r_i)[I_{\{r^{\varepsilon}(t+t_2)=r_i\}}-\nu_i]+a\sum_{i=2}^m\nu_i\left|x^{\varepsilon}(\omega_0,t+t_2-r_i)\right|\\ &\leqslant -a\sum_{i=1}^m x^{\varepsilon}(\omega_0,t+t_2-r_i)[I_{\{r^{\varepsilon}(t+t_2)=r_i\}}-\nu_i]+a\phi\sum_{i=2}^m\nu_i \min\big\{1,a(r_i-t)\big\}. \qquad (3.23)\end{aligned}$$

Similarly, for $t\in[r_j,r_{j+1})$, $j=1,2,\dots,m-2$, since all $x^{\varepsilon}(\omega_0,t+t_2-r_1),\dots,x^{\varepsilon}(\omega_0,t+t_2-r_j)\geqslant 0$, we have

$$\begin{aligned}\dot{x}^{\varepsilon}(\omega_0,t+t_2) &\leqslant -a\sum_{i=1}^m x^{\varepsilon}(\omega_0,t+t_2-r_i)[I_{\{r^{\varepsilon}(t+t_2)=r_i\}}-\nu_i]\\ &\quad + a\phi\sum_{i=j+1}^m\nu_i \min\big\{1,a(r_i-t)\big\}. \qquad (3.24)\end{aligned}$$

Finally, for all $t \in [r_{m-1}, t_4 - t_2]$, since all $x^\varepsilon(\omega_0, t + t_2 - r_1), \dots, x^\varepsilon(\omega_0, t + t_2 - r_{m-1}) \geqslant 0$,

$$\begin{aligned}\dot{x}^\varepsilon(\omega_0, t + t_2) \leqslant{}& -a \sum_{i=1}^m x^\varepsilon(\omega_0, t + t_2 - r_i)[I_{\{r^\varepsilon(t+t_2)=r_i\}} - \nu_i] \\ &+ a\phi\nu_m \min\{1, a(r_m - t)\}.\end{aligned}$$

Noting that $x^\varepsilon(\omega_0, t_2) = 0$, we therefore have

$$\begin{aligned}x^\varepsilon(\omega_0, t_4) &= x^\varepsilon(\omega_0, t_4) - x^\varepsilon(\omega_0, t_2) = \int_0^{t_4-t_2} \dot{x}^\varepsilon(\omega_0, t_2 + s)\, ds \\ &\leqslant -a \sum_{i=1}^m \int_0^{t_4-t_2} x^\varepsilon(\omega_0, t + t_2 - r_i)[I_{\{r^\varepsilon(t+t_2)=r_i\}} - \nu_i]\, dt \\ &\quad + a\phi \sum_{i=1}^{m-1} \nu_i \int_0^{r_i} \min\{1, a(r_i - s)\}\, ds + a\phi\nu_m \int_0^{t_4-t_2} \min\{1, a(r_m - s)\}\, ds.\end{aligned}$$

Noting that $t_4 - t_2 < r_m$, it follows that

$$\begin{aligned}x^\varepsilon(\omega_0, t_4) <{}& -a \sum_{i=1}^m \int_0^{t_4-t_2} x^\varepsilon(\omega_0, t + t_2 - r_i)[I_{\{r^\varepsilon(t+t_2)=r_i\}} - \nu_i]\, dt \\ &+ a\phi \sum_{i=1}^m \nu_i \int_0^{r_i} \min\{1, a(r_i - s)\}\, ds.\end{aligned}$$

In view of Lemma 3.6,

$$\lim_{\varepsilon \to 0} \int_0^{t_4-t_2} x^\varepsilon(\omega_0, t + t_2 - r_i)[I_{\{r^\varepsilon(t+t_2)=r_i\}} - \nu_i]\, dt = 0,$$

which implies that for sufficiently small ε,

$$x^\varepsilon(\omega_0, t_4) < a\phi \sum_{i=1}^m \nu_i \int_0^{r_i} \min\{1, a(r_i - s)\}\, ds. \tag{3.25}$$

Step 2: Division into three cases.

Case 1. When $a\bar{r} \leqslant 1$, it is obvious that

$$x^\varepsilon(\omega_0, t_4) < a\phi \sum_{i=1}^m \nu_i r_i = a\bar{r}\phi \leqslant \phi,$$

which is a contradiction to the definition of ϕ in (3.19).

Case 2. When $r_1, \dots, r_j < \frac{1}{a}$, $r_{j+1}, \dots, r_m \geqslant \frac{1}{a}$, and $\frac{a^2}{2}\sum_{i=1}^{j} \nu_i r_i^2 + a\sum_{i=j+1}^{m} \nu_i (r_i - \frac{1}{2a}) \leqslant 1$,

$$
\begin{aligned}
x^{\varepsilon}(\omega_0, t_4) &< a\phi \sum_{i=1}^{j} \nu_i \int_0^{r_i} a(r_i - s)\, ds + a\phi \sum_{i=j+1}^{m} \nu_i \left[\int_0^{r_i - \frac{1}{a}} ds + \int_{r_i - \frac{1}{a}}^{r_i} a(r_i - s)\, ds \right] \\
&= a\phi \sum_{i=1}^{j} \frac{a r_i^2}{2} \nu_i + a\phi \sum_{i=j+1}^{m} \nu_i \left(r_i - \frac{1}{2a} \right) \\
&\leqslant \phi,
\end{aligned}
$$

which is also a contradiction to (3.19).

Case 3. When all $r_1, \dots, r_m \geqslant \frac{1}{a}$ and $a\bar{r} \leqslant \frac{3}{2}$,

$$
\begin{aligned}
x^{\varepsilon}(\omega_0, t_4) &< a\phi \sum_{i=1}^{m} \nu_i \left[\int_0^{r_i - \frac{1}{a}} ds + \int_{r_i - \frac{1}{a}}^{r_i} a(r_i - s)\, ds \right] \\
&= a\phi \sum_{i=1}^{m} \nu_i r_i - \frac{\phi}{2} \\
&= \phi \left(a\bar{r} - \frac{1}{2} \right) \\
&\leqslant \phi,
\end{aligned}
$$

which is again a contradiction to (3.19). All of these three cases yield the contradiction. Thus, the proof is now complete. □

Proof of Theorem 3.1. If we can prove that for any initial data $\xi \in C^1([-r_m, 0]; \mathbb{R})$, the solution of Eq. (2.4) satisfies that for all $t \geqslant -r_m$,

$$
\lim_{\varepsilon \to 0} \left| x^{\varepsilon}(t) \right| \leqslant 2\|\xi\| e^{2ar_m} =: \psi_0, \quad \text{a.s.} \tag{3.26}
$$

then the almost sure uniform stability as $\varepsilon \to 0$ follows. Suppose that (3.26) is not true, namely, there exists $A \subset \Omega$ with $\mathbb{P}(A) > 0$ such that for each $\omega \in A$, $\lim_{\varepsilon \to 0} |x^{\varepsilon}(\omega, t)| \leqslant \psi_0$ does not hold. There must exist $t_4 > 0$ such that $\lim_{\varepsilon \to 0} |x^{\varepsilon}(\omega, t_4)| > \psi_0$. This implies that there exists sufficiently small ε_0 such that for any $\varepsilon \in (0, \varepsilon_0]$ such that $|x^{\varepsilon}(\omega, t_4)| > \psi_0$. Note that $x^{\varepsilon}(\omega, 0) \leqslant \|\xi\| < \psi_0$. Then we can choose t_2 and t_3 such that $0 < t_2 < t_3 \leqslant t_4$ and $|x^{\varepsilon}(\omega, t_2)| = \psi_0$ and $|x^{\varepsilon}(\omega, t_3)| > \psi_0$,

$$
\begin{cases}
\left| x^{\varepsilon}(\omega, t) \right| \leqslant \psi_0 & \text{for all } t \in [-r_m, t_2]; \\
\left| x^{\varepsilon}(\omega, t) \right| > \psi_0 & \text{for all } t \in (t_2, t_3]
\end{cases} \tag{3.27}
$$

and

$$
\frac{d}{dt}\left[x^{\varepsilon}(\omega, t) \right]^2 \Big|_{t=t_3} = 2x^{\varepsilon}(\omega, t_3)\dot{x}^{\varepsilon}(\omega, t)|_{t=t_3} > 0. \tag{3.28}
$$

By Lemma 3.4,

$$
\left| x^{\varepsilon}(\omega, t) \right| \leqslant 2\|\xi\| e^{2ar_m} \quad \text{for all } t \in [-r_m, 2r_m].
$$

Hence $2r_m < t_3$ by the definition of t_3. Then it follows from Lemma 3.7 and (3.28) that there exists $t_1 \in (t_3 - r_m, t_3)$ such that $x^\varepsilon(\omega, t_1) = 0$. By (3.27), $t_1 < t_2$ and hence $|x^\varepsilon(\omega, t)| \leqslant \psi_0$ for all $t \in [t_1 - r_m, t_1]$. Therefore, by Lemma 3.8, for all $t \in [t_1, t_3]$,

$$\lim_{\varepsilon\to 0}\left|x^\varepsilon(\omega, t)\right| \leqslant \sup_{s\in[t_1-2r_m, t_1]}\Big(\lim_{\varepsilon\to 0}\left|x^\varepsilon(\omega, s)\right|\Big) \leqslant \psi_0,$$

which yields a contradiction at $t = t_3$. This completes the proof. □

To prove the almost sure asymptotic uniform stability under $\varepsilon \to 0$, under the conditions in Theorem 3.2, let us strengthen the result of Lemma 3.8 as follows.

Lemma 3.9. *Assume that the conditions in Theorem* 3.2 *are satisfied and let $t_1 \geqslant r_m$. Let $x^\varepsilon(t)$ be a solution of* (2.4) *on $[t_1 - 2r_m, T]$ such that $T > t_1 + r_m$ and $x^\varepsilon(t_1) = 0$. Then*

$$\mathbb{P}\Big[\sup_{t\in[t_1,T]}\Big(\lim_{\varepsilon\to 0}\left|x^\varepsilon(t)\right|\Big) \leqslant \theta \sup_{t\in[t_1-2r_m, t_1]}\Big(\lim_{\varepsilon\to 0}\left|x^\varepsilon(t)\right|\Big)\Big] = 1,$$

where $\theta \in (0, 1)$ is defined as

$$\theta = \begin{cases} a\bar{r}, & 0 < a\bar{r} < 1; \\ \frac{a^2}{2}\sum_{i=1}^{j}\nu_i r_i^2 + a\sum_{i=j+1}^{m}\nu_i\big(r_i - \frac{1}{2a}\big), & r_1, \ldots, r_j < \frac{1}{a} \text{ and } r_{j+1}, \ldots, r_m \geqslant \frac{1}{a}, \\ & 0 < \frac{a^2}{2}\sum_{i=1}^{j}\nu_i r_i^2 + a\sum_{i=j+1}^{m}\nu_i\big(r_i - \frac{1}{2a}\big) < 1; \\ a\bar{r} - \frac{1}{2}, & \text{all } r_1, r_2, \ldots, r_m \geqslant \frac{1}{a} \text{ and } 0 < a\bar{r} < \frac{3}{2}. \end{cases} \tag{3.29}$$

This proof is similar to Lemma 3.8 and is thus omitted.

Proof of Theorem 3.2. Define the stopping time

$$\widetilde{T} = \sup\big\{t > 0\colon \left|x^\varepsilon(t)\right| = 0\big\}.$$

If $\widetilde{T} < \infty$, then for all $t > \widetilde{T}$, $x^\varepsilon(t) \neq 0$. By Lemma 3.7, for $t > \widetilde{T} + r_m$, $x^\varepsilon(t)$ has the same sign as $x^\varepsilon(t - r^\varepsilon(t))$, so

$$\frac{d}{dt}\big[x^\varepsilon(t)\big]^2 = -2ax^\varepsilon(t)x^\varepsilon\big(t - r^\varepsilon(t)\big) < 0 \quad \text{a.s.}$$

on $(\widetilde{T} + r_m, \infty)$, which shows that $|x^\varepsilon(t)| \to 0$ as $t \to \infty$ since 0 is the unique trivial solution of the switching system (2.4). If $\widetilde{T} = \infty$, there exists a sequence $\{\tau_i\}_{i\geqslant 0}$ with $\tau_i \to \infty$ as $i \to \infty$ such that $x^\varepsilon(\tau_i) = 0$ for each τ_i. The sequence may be chosen so that $\tau_{i+1} > \tau_i + 2r_m$. Let $\rho_i = \sup_{s\in[\tau_{i-1}, \tau_i]}(\lim_{\varepsilon\to 0}|x^\varepsilon(s)|)$ for all $i = 1, 2, \ldots$. By Lemma 3.9,

$$\theta\rho_i \geqslant \sup_{t\geqslant\tau_i}\Big(\lim_{\varepsilon\to 0}\left|x^\varepsilon(t)\right|\Big) \geqslant \sup_{t\in[\tau_i, \tau_{i+1}]}\Big(\lim_{\varepsilon\to 0}\left|x^\varepsilon(t)\right|\Big) = \rho_{i+1}, \quad \text{a.s.}$$

where $\theta \in (0, 1)$ is defined by (3.29). By induction $\rho_{n+1} \leqslant \theta^n\rho_1$, so $\rho_n \to 0$ as $n \to \infty$ since $\theta \in (0, 1)$. The above two assertions implies that for any $\eta > 0$, there exists $T_0 = T_0(\eta) > 0$ such that for all $t \geqslant T_0$ and $\|\xi\| < \delta$, $\lim_{\varepsilon\to 0}|x^\varepsilon(t)| \leqslant \eta$ a.s. Thus the proof is concluded. □

4. Stability of random delay functional systems

In this section, we extend the previous results to random delay systems of functional equations. To consider functional systems with random delay, let us present some additional notation. Let $C([-r_i,0];\mathbb{R})$ denote the family of continuous functions from $[-r_i,0]$ to $\mathbb{R}$. Consider the scalar functional system with random delay

$$\dot{x}^{\varepsilon}(t)=f\big(x_t^{r,\varepsilon}\big) \tag{4.1}$$

with the initial data $\xi\in C^1([-r_m,0];\mathbb{R})$, where $f:C([-r_m,0];\mathbb{R})\to\mathbb{R}$ is a scalar continuous functional and $x_t^{r,\varepsilon}:=x_t^{r,\varepsilon}(\theta)=\{x^{\varepsilon}(t+\theta)\colon\ -r^{\varepsilon}(t)\leqslant\theta\leqslant 0\}$. When $r(t)=r_i$, we rewrite $x_t^{r,\varepsilon}$ as $x_t^{r_i,\varepsilon}=\{x^{\varepsilon}(t+\theta)\colon\ -r_i\leqslant\theta\leqslant 0\}$.

The random delay functional system (4.1) includes the integro-differential system with random delay as a special case, for example,

$$\dot{x}^{\varepsilon}(t)=\int_{-r^{\varepsilon}(t)}^{0} f\big(x^{\varepsilon}(t+\theta)\big)\,d\mu(\theta),$$

where f is a continuous function and μ is a measure with bounded variation for every $r_i\in\mathbb{S}$, namely, $\int_{-r_i}^{0}|d\mu|<\infty$. To proceed, assume that f satisfies the Lipschitz condition and the Yorke condition; see [36].

Assumption 4.1 *(The Lipschitz condition).* Let λ be a probability measure on $[-r_m,0]$. For any $\varphi,\phi\in C([-r_m,0];\mathbb{R})$, there exists a constant $\overline{K}$ such that f satisfies

$$\big|f(\varphi)-f(\phi)\big|\leqslant\overline{K}\int_{-r_m}^{0}\big|\varphi(\theta)-\phi(\theta)\big|\,d\lambda(\theta) \tag{4.2}$$

on all $t\geqslant 0$.

In this paper, the probability measure λ may be extended to any function on $[-r_m,0]$ with bounded variation.

Assumption 4.2 *(The Yorke condition).* Assume that there exists $a>0$ such that for every $r_i\in\mathbb{S}$ and $\phi\in C([-r_i,0];\mathbb{R})$, $i=1,2,\dots,m$,

$$-aM_i(\phi)\leqslant f(\phi)\leqslant aM_i(-\phi),$$

where $M_i(\phi)=\max\{0,\sup_{s\in[-r_i,0]}\phi(s)\}$.

Note that the Yorke condition shows that f satisfies the linear growth condition and the zero solution is the unique trivial solution of the random delay functional system (4.1).

Theorem 4.1. *Let Assumptions* 4.1 *and* 4.2 *hold and suppose that one of the following conditions is satisfied*:

(i) $a\bar{r}\leqslant 1$;
(ii) $r_1,\dots,r_j<\frac{1}{a}$, $r_{j+1},\dots,r_m\geqslant\frac{1}{a}$, $\frac{a^2}{2}\sum_{i=1}^{j}\nu_i r_i^2+a\sum_{i=j+1}^{m}\nu_i(r_i-\frac{1}{2a})\leqslant 1$;
(iii) *all* $r_1,\dots,r_m\geqslant\frac{1}{a}$, $a\bar{r}\leqslant\frac{3}{2}$.

Then the trivial solution of (4.1) *is almost surely uniformly stable as* $\varepsilon\to 0$.

Theorem 4.2. *Let Assumptions* 4.1 *and* 4.2 *hold and suppose that one of the following conditions is satisfied*:

(i) $0 < a\bar{r} < 1$;
(ii) $r_1, \ldots, r_j < \frac{1}{a}$, $r_{j+1}, \ldots, r_m \geqslant \frac{1}{a}$, $0 < \frac{a^2}{2}\sum_{i=1}^{j}\nu_i r_i^2 + a\sum_{i=j+1}^{m}\nu_i(r_i - \frac{1}{2a}) < 1$;
(iii) *all* $r_1, \ldots, r_m \geqslant \frac{1}{a}$, $0 < a\bar{r} < \frac{3}{2}$.

Then the trivial solution of (4.1) *is almost surely asymptotically uniformly stable as* $\varepsilon \to 0$.

Since Assumption 4.2 shows that (4.1) satisfies the linear growth condition, so Lemma 3.4 still holds for the system (4.1). Let $x^\varepsilon(t)$ be the solution of the random delay functional system (4.1). For $r_i^\varepsilon \in \mathbb{S}$ and any $t \in [0, T]$, define a sequence of weighted occupation measures $\overline{Z}_i^\varepsilon(t)$ as follows:

$$\overline{Z}_i^\varepsilon(t) = \int_0^t f\left(x_s^{r_i,\varepsilon}\right)[I_{r^\varepsilon(s)=r_i} - \nu_i]\,ds. \tag{4.3}$$

Under Assumptions 4.1 and 4.2, we establish the following lemma similar to Lemma 3.5.

Lemma 4.3. *Let Assumptions* 4.1 *and* 4.2 *hold and* $\overline{Z}_i^\varepsilon(t)$ *be defined by* (4.3)*. Then*

$$\lim_{\varepsilon\to 0}\Big(\sup_{0\leqslant t\leqslant T}\mathbb{E}\big|\overline{Z}_i^\varepsilon(t)\big|^2\Big) = 0. \tag{4.4}$$

Proof. Define the same partition of $[0, t]$ as the proof of Lemma 3.5 and consider the piecewise functional:

$$\tilde{f}(t) = \begin{cases} f(x_0^{r_i,\varepsilon}), & \text{if } 0 \leqslant t < t_2; \\ f(x_{t_{k-1}}^{r_i,\varepsilon}), & \text{if } t_k \leqslant t < t_{k+1}; \\ f(x_{t_{N-1}}^{r_i,\varepsilon}), & \text{if } t = t_{N+1}. \end{cases}$$

Using techniques similar to that of (3.3) gives

$$\begin{aligned}\mathbb{E}\big|\overline{Z}_i^\varepsilon(t)\big|^2 &\leqslant 2\mathbb{E}\left|\int_0^t \left[f\left(x_s^{r_i,\varepsilon}\right) - \tilde{f}(s)\right][I_{\{r^\varepsilon(s)=r_i\}} - \nu_i]\,ds\right|^2 \\ &\quad + 2\mathbb{E}\left|\int_0^t \tilde{f}(s)[I_{\{r^\varepsilon(s)=r_i\}} - \nu_i]\,ds\right|^2.\end{aligned} \tag{4.5}$$

The Cauchy–Schwarz inequality yields that for any $t \in [0, T]$,

$$\mathbb{E}\left|\int_0^t \left[f\left(x_s^{r_i,\varepsilon}\right) - \tilde{f}(s)\right][I_{\{r^\varepsilon(s)=r_i\}} - \nu_i]\,ds\right|^2 \leqslant T\int_0^t \mathbb{E}\left[f\left(x_s^{r_i,\varepsilon}\right) - \tilde{f}(s)\right]^2 ds.$$

For $\tau \in [0, t_2]$, Assumption 4.1 gives

$$\int_0^\tau \mathbb{E}\big[f\big(x_s^{r_i,\varepsilon}\big) - \tilde{f}(s)\big]^2 ds = \int_0^\tau \mathbb{E}\big[f\big(x_s^{r_i,\varepsilon}\big) - f\big(x_0^{r,\varepsilon}\big)\big]^2 ds$$

$$\leqslant \overline{K}^2 \int_0^\tau \mathbb{E}\Bigg[\int_{-r_m}^0 \big(x^\varepsilon(s+\theta) - \xi(\theta)\big)\, d\lambda(\theta)\Bigg]^2 ds.$$

Note that λ is a probability measure. Applying the Hölder inequality gives

$$\Bigg[\int_{-r_m}^0 \big(x^\varepsilon(s+\theta) - \xi(\theta)\big)\, d\lambda(\theta)\Bigg]^2 \leqslant \int_{-r_m}^0 \big[x^\varepsilon(s+\theta) - \xi(\theta)\big]^2\, d\lambda(\theta).$$

We therefore have

$$\int_0^\tau \mathbb{E}\big[f\big(x_s^{r_i,\varepsilon}\big) - \tilde{f}(s)\big]^2 ds$$

$$\leqslant \overline{K}^2 \int_0^\tau \int_{-r_m}^0 \mathbb{E}\big(x^\varepsilon(s+\theta) - \xi(\theta)\big)^2\, d\lambda(\theta)\, ds$$

$$= \overline{K}^2 \iint_{D_1} \mathbb{E}\big(\xi(s+\theta) - \xi(\theta)\big)^2\, d\lambda(\theta)\, ds + \overline{K}^2 \iint_{D_2} \mathbb{E}\big(x^\varepsilon(s+\theta) - \xi(\theta)\big)^2\, d\lambda(\theta)\, ds, \tag{4.6}$$

where $D_1 = \{(s,\theta)\colon\ s \in [0,\tau],\ \theta \in [-r_m, 0],\ s+\theta < 0\}$ and $D_2 = \{(s,\theta)\colon\ s \in [0,\tau],\ \theta \in [-r_m, 0],\ s+\theta \geqslant 0\}$. Recalling that the initial data satisfies the Lipschitz condition, we have

$$\iint_{D_1} \mathbb{E}\big(\xi(s+\theta) - \xi(\theta)\big)^2\, d\lambda(\theta)\, ds \leqslant K^2 \iint_{D_1} s^2\, d\lambda(\theta)\, ds$$

$$\leqslant K^2 \int_0^\tau s^2\, ds \int_{-r_m}^0 d\lambda(\theta)$$

$$= \frac{K^2\tau^3}{3} \leqslant \frac{K^2 t_2^3}{3} = O\big(\varepsilon^{3-3\zeta}\big). \tag{4.7}$$

Noting that $s+\theta \geqslant 0$ for $(s,\theta) \in D_2$, we therefore have

$$\iint_{D_2} \mathbb{E}\big(x^\varepsilon(s+\theta) - \xi(\theta)\big)^2\, d\lambda(\theta)\, ds$$

$$\leqslant 2\iint_{D_2} \mathbb{E}\big(x^\varepsilon(s+\theta) - x^\varepsilon(0)\big)^2\, d\lambda(\theta)\, ds + 2\iint_{D_2} \mathbb{E}\big(\xi(0) - \xi(\theta)\big)^2\, d\lambda(\theta)\, ds. \tag{4.8}$$

Since $0 \leqslant -\theta \leqslant s$ for $(s,\theta) \in D_2$, similar to the computation of (4.7),

$$\iint_{D_2} \mathbb{E}\big(\xi(0) - \xi(\theta)\big)^2 \, d\lambda(\theta)\, ds = O\big(\varepsilon^{3-3\zeta}\big). \tag{4.9}$$

Assumption 4.2 shows that f satisfies the linear growth condition, so, for any $t \in [0,T]$ and $r(t) \in \mathbb{S}$, so the result of Lemma 3.4 still hold for the functional system (4.1), namely, for any $t \in [0,T]$,

$$\big|f\big(x_t^{r,\varepsilon}\big)\big| \leqslant 2a\|\xi\|e^{aT}.$$

Noting that $0 \leqslant s+\theta \leqslant s$ for $(s,\theta) \in D_2$, we therefore have

$$\begin{aligned}
\iint_{D_2} \mathbb{E}\big(x^\varepsilon(s+\theta) - x^\varepsilon(0)\big)^2 \, d\lambda(\theta)\, ds &= \iint_{D_2} \mathbb{E}\left|\int_0^{s+\theta} \dot{x}^\varepsilon(\varsigma)\, d\varsigma\right|^2 d\lambda(\theta)\, ds \\
&= \iint_{D_2} \mathbb{E}\left|\int_0^{s+\theta} f\big(x_\varsigma^{r_i,\varepsilon}\big)\, d\varsigma\right|^2 d\lambda(\theta)\, ds \\
&\leqslant 4a^2\|\xi\|^2 e^{2aT} \iint_{D_2} (s+\theta)^2 \, d\lambda(\theta)\, ds \\
&\leqslant 4a^2\|\xi\|^2 e^{2aT} \int_0^\tau s^2\, ds \int_{-r_m}^0 d\lambda(\theta) \\
&= \frac{4}{3} a^2 \|\xi\|^2 e^{2aT} \tau^3 = O\big(\varepsilon^{3-3\zeta}\big).
\end{aligned} \tag{4.10}$$

Substituting (4.7), (4.9) and (4.10) into (4.6) gives

$$\int_0^{t_2} \mathbb{E}\big[f\big(x_s^{r_i,\varepsilon}\big) - \tilde{f}(s)\big]^2 \, ds = O\big(\varepsilon^{3-3\zeta}\big).$$

This together with Assumption 4.1 gives

$$\begin{aligned}
&\int_0^t \mathbb{E}\big|f\big(x_s^{r_i,\varepsilon} - \tilde{f}(s)\big)\big|^2 \, ds \\
&\quad = \sum_{k=2}^N \int_{t_k}^{t_{k+1}} \mathbb{E}\big|f(x_s^{r_i,\varepsilon} - f\big(x_{t_{k-1}}^{r_i,\varepsilon}\big)\big|^2 \, ds + O\big(\varepsilon^{3-3\zeta}\big) \\
&\quad \leqslant \overline{K}^2 \sum_{k=2}^N \int_{t_k}^{t_{k+1}} \int_{-r_m}^0 \mathbb{E}\big|x^\varepsilon(s+\theta) - x^\varepsilon(t_{k-1}+\theta)\big|^2 \, d\lambda(\theta)\, ds + O\big(\varepsilon^{3-3\zeta}\big).
\end{aligned}$$

Now let us estimate

$$\int_{t_k}^{t_{k+1}} \int_{-r_m}^{0} \mathbb{E}\left|x^\varepsilon(s+\theta) - x^\varepsilon(t_{k-1}+\theta)\right|^2 d\lambda(\theta)\, ds.$$

If $t_{k-1} \geqslant r_m$, similar to the computation of (4.10),

$$\begin{aligned}
&\int_{t_k}^{t_{k+1}} \int_{-r_m}^{0} \mathbb{E}\left|x^\varepsilon(s+\theta) - x^\varepsilon(t_{k-1}+\theta)\right|^2 d\lambda(\theta)\, ds \\
&\quad = \int_{t_k}^{t_{k+1}} \int_{-r_m}^{0} \mathbb{E}\left| \int_{t_{k-1}+\theta}^{s+\theta} \dot{x}^\varepsilon(\varsigma)\, d\varsigma \right|^2 d\lambda(\theta)\, ds \\
&\quad = \int_{t_k}^{t_{k+1}} \int_{-r_m}^{0} \mathbb{E}\left| \int_{t_{k-1}+\theta}^{s+\theta} f\left(x_s^{r_i,\varepsilon}\right) d\varsigma \right|^2 d\lambda(\theta)\, ds \\
&\quad \leqslant 4a^2 \|\xi\|^2 e^{2aT} \int_{t_k}^{t_{k+1}} (s - t_{k-1})^2\, ds \\
&\quad \leqslant 4a^2 \|\xi\|^2 e^{2aT} (t_{k+1} - t_{k-1})^3 \\
&\quad = O\left(\varepsilon^{3-3\zeta}\right).
\end{aligned} \tag{4.11}$$

If $t_{k-1} < r_m$,

$$\begin{aligned}
&\int_{t_k}^{t_{k+1}} \int_{-r_m}^{0} \mathbb{E}\left|x^\varepsilon(s+\theta) - x^\varepsilon(t_{k-1}+\theta)\right|^2 d\lambda(\theta)\, ds \\
&\quad = \int_{t_k}^{t_{k+1}} \int_{-r_m}^{-t_{k-1}} \mathbb{E}\left|x^\varepsilon(s+\theta) - \xi(t_{k-1}+\theta)\right|^2 d\lambda(\theta)\, ds \\
&\qquad + \int_{t_k}^{t_{k+1}} \int_{-t_{k-1}}^{0} \mathbb{E}\left|x^\varepsilon(s+\theta) - x^\varepsilon(t_{k-1}+\theta)\right|^2 d\lambda(\theta)\, ds.
\end{aligned}$$

Similar to the computation of (4.11),

$$\int_{t_k}^{t_{k+1}} \int_{-t_{k-1}}^{0} \mathbb{E}\left|x^\varepsilon(s+\theta) - x^\varepsilon(t_{k-1}+\theta)\right|^2 d\lambda(\theta)\, ds = O\left(\varepsilon^{3-3\zeta}\right). \tag{4.12}$$

Define $D_3 = \{(s,\theta)\colon\ s \in [t_k, t_{k+1}],\ \theta \in [-r_m, -t_{k-1}],\ s+\theta \geqslant 0\}$ and $D_4 = \{(s,\theta)\colon\ s \in [t_k, t_{k+1}],\ \theta \in [-r_m, -t_{k-1}],\ s+\theta < 0\}$. Then

$$\begin{aligned}
&\int_{t_k}^{t_{k+1}} \int_{-r_m}^{-t_{k-1}} \mathbb{E}\left|x^{\varepsilon}(s+\theta) - \xi(t_{k-1}+\theta)\right|^2 d\lambda(\theta)\, ds \\
&\quad = \iint_{D_3} \mathbb{E}\left|x^{\varepsilon}(s+\theta) - \xi(t_{k-1}+\theta)\right|^2 d\lambda(\theta)\, ds \\
&\qquad + \iint_{D_4} \mathbb{E}\left|\xi(s+\theta) - \xi(t_{k-1}+\theta)\right|^2 d\lambda(\theta)\, ds.
\end{aligned}$$

Similar to the computation of (4.7),

$$\iint_{D_4} \mathbb{E}\left|\xi(s+\theta) - \xi(t_{k-1}+\theta)\right|^2 d\lambda(\theta)\, ds = O\left(\varepsilon^{3-3\zeta}\right) \tag{4.13}$$

and similar to (4.8),

$$\iint_{D_3} \mathbb{E}\left|x^{\varepsilon}(s+\theta) - \xi(t_{k-1}+\theta)\right|^2 d\lambda(\theta)\, ds = O\left(\varepsilon^{3-3\zeta}\right). \tag{4.14}$$

(4.12), (4.13), and (4.14) show that if $t_{k-1} < r_m$,

$$\int_{t_k}^{t_{k+1}} \int_{-r_m}^{0} \mathbb{E}\left|x^{\varepsilon}(s+\theta) - x^{\varepsilon}(t_{k-1}+\theta)\right|^2 d\lambda(\theta)\, ds = O\left(\varepsilon^{3-3\zeta}\right).$$

This, together with (4.11), gives

$$\int_{t_k}^{t_{k+1}} \int_{-r_m}^{0} \mathbb{E}\left|x^{\varepsilon}(s+\theta) - x^{\varepsilon}(t_{k-1}+\theta)\right|^2 d\lambda(\theta)\, ds = O\left(\varepsilon^{3-3\zeta}\right),$$

which shows that

$$\int_0^t \mathbb{E}\left|f\left(x_s^{r_i,\varepsilon}\right) - \tilde{f}(s)\right|^2 ds = \overline{K}^2 \sum_{k=0}^{N} O\left(\varepsilon^{3-3\zeta}\right) = O\left(\varepsilon^{2-2\zeta}\right). \tag{4.15}$$

Let us now estimate the second term of (4.5). Denote

$$\overline{\eta}^{\varepsilon}(t) = \mathbb{E}\left[\int_0^t \tilde{f}(s)(I_{\{r^{\varepsilon}(s)=r_i\}} - \nu_i)\, ds\right]^2.$$

Note that for all $s \in [0, T]$, $\tilde{f}(s)$ is a bounded functional. When $t \in [t_k, t_{k+1})$, $\tilde{f}$ is $\mathcal{F}_{t_{k-1}}$-measurable. Hence, repeating the same proof process of Lemma 3.5, similar to (3.9), we may obtain

$$\sup_{0\leqslant t\leqslant T} \overline{\eta}^{\varepsilon}(t) = O\left(\varepsilon^{1-\zeta}\right).$$

This, together with (4.15) gives the desired assertion (4.4). □

Using similar technique to Lemma 3.6, we can establish the following convergence result of $\overline{Z}_i^{\varepsilon}$.

Lemma 4.4. *Let $\overline{Z}_i^{\varepsilon}$ be defined by* (4.3)*. For any $t \in [0, T]$,*

$$\mathbb{P}\Big(\lim_{\varepsilon\to 0}\big|\overline{Z}_i^{\varepsilon}(t)\big| = 0\Big) = 1.$$

Note that Lemma 3.7 still holds for system (4.1). Now let us give the result similar to Lemma 3.8 for the random delay functional system (4.1).

Lemma 4.5. *Assume that the conditions in Theorem* 4.1 *are satisfied and let $t_1 \geqslant r_m$. Let $x^{\varepsilon}(t)$ be a solution of* (4.1) *on $[t_1 - 2r_m, T]$ such that $T > t_1 + r_m$ and $x^{\varepsilon}(t_1) = 0$. Then*

$$\mathbb{P}\Big[\sup_{t\in[t_1,T]}\Big(\lim_{\varepsilon\to 0}\big|x^{\varepsilon}(t)\big|\Big) \leqslant \sup_{s\in[t_1-2r_m,t_1]}\Big(\lim_{\varepsilon\to 0}\big|x^{\varepsilon}(s)\big|\Big)\Big] = 1. \tag{4.16}$$

Proof. The proof is similar to that of Lemma 3.8. Suppose that the assertion (4.16) is not true. Then there exists $A \subset \Omega$ with $\mathbb{P}(A) > 0$ such that for each $\omega \in A$

$$\sup_{t\in[t_1,T]}\Big(\lim_{\varepsilon\to 0}\big|x^{\varepsilon}(\omega,t)\big|\Big) > \sup_{s\in[t_1-2r_m,t_1]}\Big(\lim_{\varepsilon\to 0}\big|x^{\varepsilon}(\omega,s)\big|\Big).$$

By the uniform boundedness of $x^{\varepsilon}(t)$ on $t \in [0, T]$, there exists sufficiently small ε_0 such that for any $\varepsilon \in (0, \varepsilon_0]$,

$$\sup_{t\in[t_1,T]}\big|x^{\varepsilon}(\omega,t)\big| > \sup_{s\in[t_1-2r_m,t_1]}\big|x^{\varepsilon}(\omega,s)\big|.$$

Similar to the proof of Lemma 3.8, let us fix $\omega_0 \in A$ and use the same definitions of ϕ, ϕ_0, t_2, t_3 and t_4 as before. By Assumption 4.2, for all $t \in [t_1 - r_m, t_4]$,

$$\big|\dot{x}^{\varepsilon}(\omega_0,t)\big| = \big|f\big(x_{t,\omega_0}^{r,\varepsilon}\big)\big| \leqslant a\Big(\sup_{-r_m\leqslant\theta\leqslant 0}\big|x(\omega_0,t+\theta)\big|\Big) \leqslant a\phi, \tag{4.17}$$

which, together with $x^{\varepsilon}(\omega_0, t_2) = 0$, gives

$$\begin{aligned}\big|x^{\varepsilon}(\omega_0,t)\big| &= \big|x^{\varepsilon}(\omega_0,t_2) - x^{\varepsilon}(\omega_0,t)\big| = \left|\int_t^{t_2} f\big(x_{s,\omega_0}^{r,\varepsilon}\big)\,ds\right| \\ &\leqslant \left|\int_t^{t_2} a\Big[\sup_{-r_m\leqslant\theta\leqslant 0}\big|x^{\varepsilon}(\omega_0,s+\theta)\big|\Big]\,ds\right| \leqslant a\phi|t_2 - t|, \end{aligned}\tag{4.18}$$

where $x_{t,\omega_0}^{r,\varepsilon} = \{x(\omega_0, t+\theta),\ -r(t) \leqslant \theta \leqslant 0\}$. Similar to (3.22), it follows from (4.17) and (4.18) that for $t_2 \leqslant t + t_2 \leqslant \min\{t_2 + r_1, t_4\}$,

$$\begin{aligned}
\dot{x}^{\varepsilon}(\omega_0, t+t_2) &= \sum_{i=1}^{m} f\big(x_{t+t_2,\omega_0}^{r_i,\varepsilon}\big) I_{\{r^{\varepsilon}(t+t_2)=r_i\}} \\
&= \sum_{i=1}^{m} f\big(x_{t+t_2,\omega_0}^{r_i,\varepsilon}\big)[I_{\{r^{\varepsilon}(t+t_2)=r_i\}} - \nu_i] - a \sum_{i=1}^{m} \nu_i x^{\varepsilon}(\omega_0, t+t_2-r_i) \\
&\leqslant \sum_{i=1}^{m} f\big(x_{t+t_2,\omega_0}^{r_i,\varepsilon}\big)[I_{\{r^{\varepsilon}(t+t_2)=r_i\}} - \nu_i] + a\phi \sum_{i=1}^{m} \nu_i \min\big\{1, a(r_i - t)\big\}.
\end{aligned}$$

Note that $t \in [r_1, t_4 - t_2]$ is equivalent to $t + t_2 - r_1 \in [t_2, t_4 - r_1]$, so $x^{\varepsilon}(\omega_0, t+t_2-r_1) \geqslant 0$. Similar to (3.23), we have that for all $t \in [r_1, r_2)$,

$$\dot{x}^{\varepsilon}(\omega_0, t+t_2) \leqslant \sum_{i=1}^{m} f\big(x_{t+t_2,\omega_0}^{r_i,\varepsilon}\big)[I_{\{r^{\varepsilon}(t+t_2)=r_i\}} - \nu_i] + a\phi \sum_{i=2}^{m} \nu_i \min\big\{1, a(r_i - t)\big\}.$$

Similarly, for $t \in [r_j, r_{j+1})$, $j = 1, 2, \ldots, m-2$, since all $x^{\varepsilon}(\omega_0, t+t_2-r_1), \ldots, x^{\varepsilon}(\omega_0, t+t_2-r_j) \geqslant 0$, we have

$$\dot{x}^{\varepsilon}(\omega_0, t+t_2) \leqslant \sum_{i=1}^{m} f\big(x_{t+t_2,\omega_0}^{r_i,\varepsilon}\big)[I_{\{r^{\varepsilon}(t+t_2)=r_i\}} - \nu_i] + a\phi \sum_{i=j+1}^{m} \nu_i \min\big\{1, a(r_i - t)\big\}.$$

Finally, for all $t \in [r_{m-1}, t_4 - t_2]$, since all $x^{\varepsilon}(\omega_0, t+t_2-r_1), \ldots, x^{\varepsilon}(\omega_0, t+t_2-r_{m-1}) \geqslant 0$, similar to (3.24), we have

$$\dot{x}^{\varepsilon}(\omega_0, t+t_2) \leqslant \sum_{i=1}^{m} f\big(x_{t+t_2,\omega_0}^{r_i,\varepsilon}\big)[I_{\{r^{\varepsilon}(t+t_2)=r_i\}} - \nu_i] + a\phi \nu_m \min\big\{1, a(r_m - t)\big\}.$$

Noting that $x^{\varepsilon}(\omega_0, t_2) = 0$, we therefore have

$$\begin{aligned}
x^{\varepsilon}(\omega_0, t_4) &= x^{\varepsilon}(\omega_0, t_4) - x^{\varepsilon}(\omega_0, t_2) \\
&= \int_0^{t_4-t_2} \dot{x}^{\varepsilon}(\omega_0, t_2+s)\,ds \\
&\leqslant \sum_{i=1}^{m} \int_0^{t_4-t_2} f\big(x_{t+t_2,\omega_0}^{r_i,\varepsilon}\big)[I_{\{r^{\varepsilon}(t+t_2)=r_i\}} - \nu_i]\,dt \\
&\quad + a\phi \sum_{i=1}^{m-1} \nu_i \int_0^{r_i} \min\big\{1, a(r_i - s)\big\}\,ds + a\phi \nu_m \int_0^{t_4-t_2} \min\big\{1, a(r_m - s)\big\}\,ds.
\end{aligned}$$

Recalling that $t_4 - t_2 < r_m$, it follows that

$$x^{\varepsilon}(\omega_0, t_4) < \sum_{i=1}^{m} \int_0^{t_4-t_2} f\big(x_{t+t_2,\omega_0}^{r_i,\varepsilon}\big)[I_{\{r^{\varepsilon}(t+t_2)=r_i\}} - \nu_i]\,dt + a\phi \sum_{i=1}^{m} \nu_i \int_0^{r_i} \min\big\{1, a(r_i - s)\big\}\,ds.$$

In view of Lemma 4.4,

$$\lim_{\varepsilon\to 0}\int_0^{t_4-t_2} f\left(x^{r_i,\varepsilon}_{t+t_2,\omega_0}\right)[I_{\{r^\varepsilon(t+t_2)=r_i\}}-\nu_i]\,dt=0,$$

which implies that for sufficiently small ε,

$$x^\varepsilon(\omega_0,t_4)<a\phi\sum_{i=1}^m \nu_i\int_0^{r_i}\min\left\{1,a(r_i-s)\right\}ds.$$

Then repeating the argument as in Lemma 3.8 leads to the desired assertion. □

Similarly, we may establish the result similar to Lemma 3.9 for the random delay functional system (4.1). Similar to proofs of Theorems 3.1 and 3.2, Theorems 4.1 and 4.2 follow. We omit the details of proofs.

5. An illustrative example

This section presents an example to demonstrates our results. Consider the scalar pure random delay system

$$\dot{x}^\varepsilon(t)=-x^\varepsilon\left(t-r^\varepsilon(t)\right) \tag{5.1}$$

with the initial data $\xi(\theta)=2$ for $\theta\in[-2,0]$, where $r^\varepsilon(t)$ is a continuous-time Markov chain whose the state space is $\mathbb{S}=\{1,2\}$ and the generator is $(Q/\varepsilon)+Q_0$ with

$$Q=\begin{bmatrix}-0.4 & 0.4\\ 0.6 & -0.6\end{bmatrix}\quad\text{and}\quad Q_0=\begin{bmatrix}-0.2 & 0.2\\ 0.1 & -0.1\end{bmatrix}.$$

It is easy to compute that the stationary distribution is given by $\nu=(0.6,0.4)$. The system (5.1) may be seen as a switching system between the two subsystems

$$\dot{x}(t)=-x(t-1), \tag{5.2}$$

$$\dot{x}(t)=-x(t-2) \tag{5.3}$$

according to this Markov chain. By Theorem 1.2, the system (5.2) is stable but the system (5.3) is unstable. Note that $\bar{r}=0.6+0.4\times 2=1.4<3/2$. By condition (iii) of Theorem 3.2, the switching system (5.1) is almost surely asymptotically uniformly stable under $\varepsilon\to 0$. When we choose $\varepsilon=0.1$, Fig. 2 confirms our results.

In fact, when all delays $r_i\geqslant 1$, the stability condition $\bar{r}<3/2$ is very sharp. To see this, let us choose

$$Q=\begin{bmatrix}-0.8 & 0.8\\ 0.2 & -0.2\end{bmatrix}$$

and Q_0 does not change, which implies that the stationary distribution is $\nu=(0.2,0.8)$ and $\bar{r}=0.2+0.8\times 2=1.8>3/2$ (is also bigger than $\pi/2$). Choose $\varepsilon=0.1$. Fig. 3 shows that the switching system is not stable.

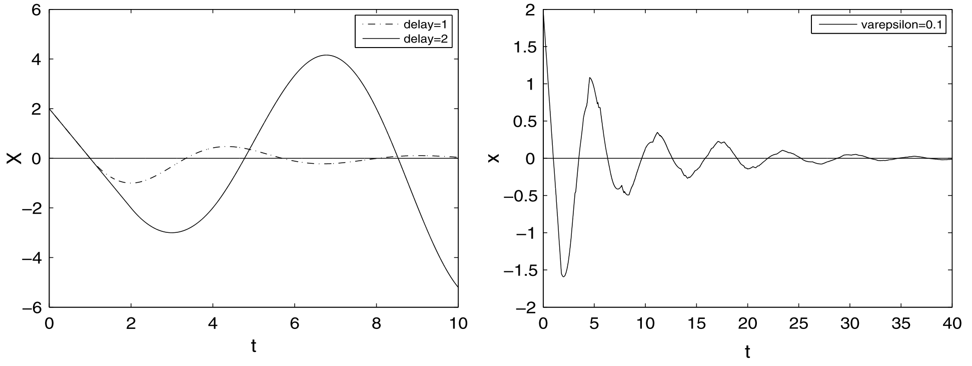

Fig. 2. Left graph: when delay is 2, the system is not stable and the system with delay 1 is stable. Right graph: $a\bar{r} = 1.4$, the switching system is stable.

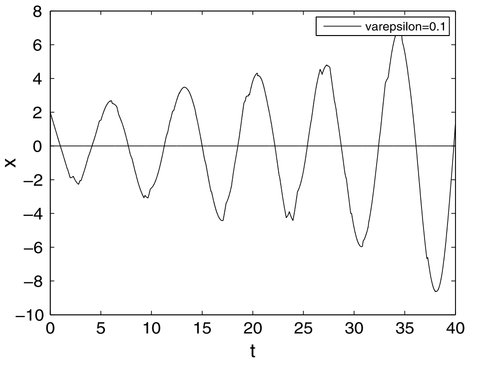

Fig. 3. $a\bar{r} = 1.8$, the switching system is not stable.

6. Concluding remarks

In this paper, we have revised Yorke's model with a substantial generalization of considering the delays being random. Our main focus has been the stability of the associated dynamic systems. Different from many existing results, the stability depends on the random delays explicitly. The results obtained will be important for systems arising in communication networks and control systems.

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References

[1] G. Badowski, G. Yin, Stability of hybrid dynamic systems containing singularly perturbed random processes, IEEE Trans. Automat. Control 47 (2002) 2021–2032.
[2] H.R. Bailey, M.Z. Williams, Some results on the differential difference equation $\dot{x}(t) = \sum_{i=0}^{N} A_i x(t - T_i)$, J. Math. Anal. Appl. 15 (1966) 569–587.
[3] H.T. Banks, Modeling and Control in the Biomedical Sciences, Lect. Notes Biomath., vol. 6, Springer-Verlag, 1975.
[4] C. Chen, S. Hirche, M. Buss, Sampled-data networked control systems with random time delay, in: Proceedings of the 17th World Congress the International Federation of Automatic Control, 2008, pp. 11594–11599.
[5] K.L. Cooke, Functional Differential Equations: Some Models and Perturbation Problems, Academic Press, 1967.

[6] J.P. Fouque, G. Papanicolaou, R.K. Sircar, Derivatives in Financial Market with Stochastic Volatility, Cambridge University Press, Cambridge, 2000.
[7] J.P. Fouque, G. Papanicolaou, R.K. Sircar, K. Solna, Multiscale stochastic volatility asymptotics, Multiscale Model. Simul. 1 (2004) 22–42.
[8] S.E. Grossman, Stability in n-dimensional differential-delay equations, J. Math. Anal. Appl. 40 (1972) 541–546.
[9] S.E. Grossman, J.A. Yorke, Asymptotic behavior and exponential stability criteria for differential delay equations, J. Differential Equations 12 (1972) 236–255.
[10] J.R. Haddock, On the asymptotic behavior of solutions of $\dot{x}(t) = -a(t) f(x(t - r(t)))$, SIAM J. Math. Anal. 5 (1974) 569–573.
[11] A. Halanay, J.A. Yorke, Some new results and problems in the theory of differential-delay equations, SIAM Rev. 13 (1971) 55–80.
[12] J.K. Hale, S.M.V. Lunel, Introduction to Functional Differential Equations, Springer-Verlag, New York, 1993.
[13] D. Huang, S.K. Nguang, State feedback control of uncertain networked control systems with random time delays, IEEE Trans. Automat. Control 53 (2008) 829–834.
[14] J. Kato, On Liapunov–Razumikhin type theorems for functional differential equations, Funkcial. Ekvac. 16 (1973) 225–239.
[15] R.Z. Khasminskii, G. Yin, On averaging principle: an asymptotic expansion approach, SIAM J. Math. Anal. 35 (2004) 1534–1560.
[16] L.B. Knyazhishche, On the 3/2 condition for the stability of solutions of scalar delay equations, Differ. Equ. 42 (2006) 188–192.
[17] V.B. Kolmanovskii, V.R. Nosov, Stability of Functional Differential Equations, Academic Press, New York, 1986.
[18] V.B. Kolmanovskii, Mean square stability of difference equations with a stochastic delay, Nonlinear Anal. 52 (2003) 795–804.
[19] I. Kolmanovsky, T.L. Maizenberg, Mean-square stability of nonlinear systems with time-varying, random delay, Stoch. Anal. Appl. 19 (2001) 279–293.
[20] T. Krisztin, Stability for functional differential equations and some variational problems, Tohoku Math. J. 42 (1990) 407–417.
[21] A. Kumar, D. Manjunath, J. Kuri, Communication Networking: An Analytical Approach, Morgan Kaufmann Publishers, 2004.
[22] G. Ladas, Y.G. Sficas, I.P. Stavroulakis, Asymptotic behavior of solutions of retarded differential equations, Proc. Amer. Math. Soc. 88 (1983) 247–253.
[23] H. Matsunaga, Exact stability criteria for delay differential and difference equations, Appl. Math. Lett. 20 (2007) 183–188.
[24] L. Montestruque, P. Antsaklis, Stability of model-based networked control systems with time-varying transmission times, IEEE Trans. Automat. Control 49 (2004) 1562–1572.
[25] J. Nilsson, B. Bernhardsson, B. Wittenmark, Stochastic analysis and control of real-time systems with random time delays, Automatica 34 (1998) 57–64.
[26] L. Schenato, Optimal estimation in networked control systems subject to random delay and packet drop, IEEE Trans. Automat. Control 53 (2008) 1311–1317.
[27] B. Sklar, Digital Communications: Fundamentals and Applications, 2nd edition, Prentice Hall, 2001.
[28] J.W.-H. So, J.S. Yu, M.-P. Chen, Asymptotic stability for scalar delay differential equations, Funkcial. Ekvac. 39 (1996) 1–17.
[29] G. Yin, Q. Zhang, Continuous-Time Markov Chains and Applications: A Singular Perturbation Approach, Springer-Verlag, New York, 1998.
[30] G. Yin, Asymptotic expansions of option price under regime-switching diffusions with a fast-varying switching process, Asymptot. Anal. 65 (2009) 203–222.
[31] T. Yoneyama, On the stability for the delay-differential equation $\dot{x}(t) = -a(t) f(x(t - r(t)))$, J. Math. Anal. Appl. 120 (1986) 271–275.
[32] T. Yoneyama, On the 3/2 stability theorem for one-dimensional delay-differential equations, J. Math. Anal. Appl. 125 (1987) 161–173.
[33] T. Yoneyama, J. Sugie, On the stability region of differential equations with two delays, Funkcial. Ekvac. 31 (1988) 233–240.
[34] T. Yoneyama, Uniform stability for one-dimensional delay-differential equations with dominant delayed term, Tohoku Math. J. 41 (1989) 217–236.
[35] T. Yoneyama, The 3/2 stability theorem for one-dimensional delay-differential equations with unbounded delay, J. Math. Anal. Appl. 165 (1992) 133–143.
[36] J.A. Yorke, Asymptotic stability for one dimensional differential-delay equations, J. Differential Equations 7 (1970) 189–202.